Exploring Symmetry-Breaking Parameters in Selected Graph Families

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Extended Abstract

The distinguishing number of a graph G is the smallest number of colors needed to label the vertices of $V(G)$ such that no non-trivial automorphism of the graph G preserves all the color classes. The distinguishing number of a graph G is denoted by $D(G)$. The concept of distinguishing coloring was introduced by Albertson in [1] and there after many researchers have worked on the problems related to the distinguishing coloring [2, 10, 13, 18] and its variants [8, 15, 20, 22, 24]. A determining set of the graph G is a subset, $S \subseteq V(G)$, such that every automorphism of G is uniquely determined by its action on S. The determining number of a graph G is the size of a smallest determining set, and it is denoted by $Det(G)$. Recently, D.L. Boutin has introduced two additional parameters related to the distinguishing coloring called paint cost and frugal distinguishing coloring [14]. For any $d \geq D(G)$, the paint cost of a d-distinguishing coloring of G, represented by $\rho^d(G)$, is defined as the smallest possible size of complement of a color class over all d-distinguishing colorings of G. The frugal distinguishing number of a graph G, denoted by $Fdist(G)$, is the smallest d for which $\rho^d(G)$ equals the determining number $Det(G)$.

It is known that $D(G) \leq Fdist(G) \leq Det(G)+1$ [14]. D.L. Boutin raised the problem of characterizing graphs for which $D(G) = Fdist(G)$. To address this question, we study the relation between the frugal distinguishing number and the distinguishing number for certain undirected Cayley graphs and Levi graphs. An *undirected Cayley graph* $\mathrm{Cay}(\Gamma, S)$ of a group Γ with respect to a generating set $S \subseteq \Gamma$ satisfying $S = S^{-1}$ is defined as the graph with vertex set Γ, where two vertices $g, h \in \Gamma$ are adjacent if and only if $g^{-1}h \in S$. In this work, we determine the distinguishing number, determining number, and frugal distinguishing number of Cayley graphs corresponding to the dihedral group D_n and the quaternion group Q_8 with specific generating sets S. A *Levi graph* is a bipartite graph associated with an incidence structure, whose vertex set consists of points and lines, with adjacency defined by incidence. In particular, we consider a family of Levi graphs where the point set consists of all singleton subsets of $[m] = \{1, 2, \ldots, m\}$ and the line set consists of all 2-element subsets of $[m]$, with adjacency defined by set containment. We compute the distinguishing number, determining number and frugal distinguishing number for these graphs.

Main results

Theorem 1. *For Cayley graph of D_n: the distinguishing number is 2 , determining number is 2, and frugal distinguishing number is 3 .*

Theorem 2. *For Cayley graph of Q_8: The distinguishing number is 5, determining number is 6, and frugal distinguishing number is 5.*

Theorem 3. *For Levi graph $L_1(m,2)$*

- *The distinguishing number $D(L_1(m,2)) = 2$,*
- *The determining number* $Det(L_1(m,2)) = \begin{cases} \frac{2(m-1)}{3}, & if \quad m-1 = 3k \\ 2\lfloor\frac{m-1}{3}\rfloor + 1, & if \quad m-1 = 3k+1 \\ 2\lfloor\frac{m-1}{3}\rfloor + 2, & if \quad m-1 = 3k+2 \end{cases}$,

- *The frugal distinguishing number* $Fdist(L_1(m,2)) = \lceil \frac{d}{2} \rceil$ *where* d *is the determining number of* $L_1(m,2)$.

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